Differential Geometry I | Week 13

Last time, we stated the following resulting, which claim that the metric tensor and the 2^{nd} fundamental form determine a connected surface in $\mathbb{R}^3$ uniquely up to a rigid motion:

<u>Theorem</u>: Let $\Omega \subseteq \mathbb{R}^2$ be connected and let $\psi : \Omega \to S \subset \mathbb{R}^3$ and $\psi' : \Omega \to S' \subset \mathbb{R}^3$ (note: same Ω!) be two C^2 parametrizations such that the corresponding matrices of the metric and the 2^{nd} fundamental form satisfy

$$g_{ij}(u) = g'_{ij}(u) \quad \text{and} \quad h_{ij}(u) = h'_{ij}(u)$$

for all $u \in \Omega$ and $i,j = 1,2$. Then there exists an (affine) isometry $F : \mathbb{R}^3 \to \mathbb{R}^3$ such that $\psi' = F \circ \psi$.

<u>Note</u>: In the above, we assume that the matrices are with respect to the associated tangent basis $b_i = \frac{\partial \psi}{\partial u_i}$ for $T_{\psi(u)}S$ (and $b'_i = \frac{\partial \psi'}{\partial u_i}$ for $T_{\psi'(u)}S'$) and co-orientation $n = \frac{b_1 \times b_2}{\|b_1 \times b_2\|}$ (and $n' = \frac{b'_1 \times b'_2}{\|b'_1 \times b'_2\|}$).

<u>Proof</u>:

For any $u \in \Omega$, let $M(u) : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map mapping the basis $\{b_1(u), b_2(u), n(u)\}$ to $\{b'_1(u), b'_2(u), n'(u)\}$,

i.e. $b'_1(u) = M(u) \cdot b_1(u)$, $b'_2(u) = M(u)\, b_2(u)$, $n'(u) = M(u) \cdot n(u)$

(note: M depends on $u \in \Omega$). We will first show that $M(u) \in SO(3)$:

- Since $g'_{ij} = g_{ij} \iff \langle b'_i, b'_j \rangle = \langle b_i, b_j \rangle$

$$\iff \langle M \cdot b_i, M \cdot b_j \rangle = \langle b_i, b_j \rangle.$$

Also $n' \perp b'_i$, $n \perp b_i$ so $\langle M b_i, M n \rangle = \langle b_i, n \rangle = 0$ for $i = 1,2$.

and $\|n'\| = 1$, $\|n\| = 1$ so $\langle Mn, Mn \rangle = \langle n, n \rangle = 1$

Combining the above: $\forall\, v \in \mathbb{R}^3$, splitting

$$v = v^{(1)} b_1 + v^{(2)} b_2 + v^{(3)} n,$$

We see that $\langle Mv, Mv \rangle = \langle v, v \rangle$, so $M(u) \in O(3)$ $\forall u \in \Omega$.

Moreover, since $n = \frac{b_1 \times b_2}{\|b_1 \times b_2\|}$ and $n' = \frac{b_1' \times b_2'}{\|b_1' \times b_2'\|}$: The bases $\{b_1, b_2, n\}$ and $\{b_1', b_2', n'\}$ are both positively oriented, so $\det(M) > 0$.

Therefore: $M(u) \in SO(3)$ $\forall u \in \Omega$

We will show that $M(u)$ is constant in u:

Note: $\frac{\partial^2 \psi'}{\partial u_i \partial u_j}(u) = \frac{\partial}{\partial u_i} b_j'(u) = \frac{\partial}{\partial u_i}\left(M(u)\, b_j(u)\right) = \frac{\partial M}{\partial u_i}(u) \cdot b_j(u) + M(u) \cdot \frac{\partial b_j}{\partial u_i}(u)$

$$= \frac{\partial M}{\partial u_i}(u) \cdot b_j(u) + M(u) \cdot \frac{\partial^2 \psi}{\partial u_i \partial u_j}(u) \quad ①$$

Moreover, since $h_{ij} = h_{ij}'$ and $g_{ij} = g_{ij}'$ (therefore also $\Gamma_{ijk} = \Gamma_{ijk}'$ since $\Gamma_{ijk} = \frac{1}{2}(\partial_i g_{kj} + \partial_j g_{ki} - \partial_k g_{ij})$), we obtain from the decomposition of $\partial^2_{ij}\psi$ into the basis $\{b_1, b_2, n\}$:

$$\frac{\partial^2 \psi}{\partial u_i \partial u_j} = \Gamma^1_{ij}\, b_1 + \Gamma^2_{ij}\, b_2 + h_{ij}\, n$$

and
$$\frac{\partial^2 \psi'}{\partial u_i \partial u_j} = (\Gamma^1_{ij})' b_1' + (\Gamma^2_{ij})' b_2' + h_{ij}' n'$$

$$= \Gamma^1_{ij} \cdot M b_1 + \Gamma^2_{ij} \cdot M b_2 + h_{ij}\, n$$

$$= M \cdot \frac{\partial^2 \psi}{\partial u_i \partial u_j} \quad ②$$

Subtracting ② from ①: We get $\frac{\partial M}{\partial u_i} b_j = 0$ for all $i, j = 1, 2$

$$\Rightarrow \frac{\partial M}{\partial u_1} = \frac{\partial M}{\partial u_2} = 0 \quad \Rightarrow M(u) \text{ is constant on } \Omega \quad (\text{since } \Omega \text{ is connected})$$

$$\text{So} \quad b_i' = M \cdot b_i \Leftrightarrow \frac{\partial \psi'}{\partial u_i}(u) = M \cdot \frac{\partial \psi}{\partial u_i}(u)$$

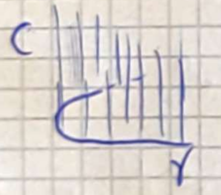
$$\Rightarrow \psi'(u) = M \cdot \psi(u) + c. \quad \boxtimes$$

It is natural to ask the following question: Are there conditions on the intrinsic geometry of a surface that make it rigid (i.e any other surface S' which is isometric to S is also related to S by a rigid motion in $\mathbb{R}^3$).

- Important in construction: Rigid surfaces do not bend.

As we have seen: In the case $K = 0$, this is not true:

Any cylinder of the form $C = \{(x,y,z) : (x,y) \in \gamma \subset \mathbb{R}^2\}$ is locally isometric to the flat plane.



However, convex compact surfaces are rigid:

Recall: A subset $\mathcal{U}$ of $\mathbb{R}^3$ is compact if and only if it is bounded and closed.

A <u>compact</u> surface S: Is sometimes also called (by abuse of terminology) a closed surface.

E.g: The sphere and the torus: Are compact.

The cylinder and the Möbius strip: Are non-compact.

(Note: A surface with "boundary" cannot be compact

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- Recall also: If $f: S \to S'$ is a homeomorphism and S is compact, then S' is compact as well
  (it is true in general: Images via continuous maps of compact sets are compact)

<u>Theorem</u> (Cohn-Vossen)
Let $S \subset \mathbb{R}^3$ be a <u>compact</u> surface with Gauss curvature $K > 0$.
If $S' \subset \mathbb{R}^3$ is any other surface such that there exists an isometry $f: S \to S'$, then f extends to an (affine) isometry $f: \mathbb{R}^3 \to \mathbb{R}^3$.

<u>Remarks</u>:
- If S is compact and $K > 0$: S is <u>necessarily convex.</u>
- For S, S' as in the assumptions of the theorem: For any local parametrization $\varphi: \Omega \to S$ and $\varphi' = f \circ \varphi: \Omega \to S'$, the expression for the metric tensors are the same $g'_{ij} = g_{ij}$. In order to show the theorem: it suffices to prove that $h'_{ij} = h_{ij}$. This follows from the fact that, when $K > 0$, the Gauss-Codazzi equations form an <u>elliptic</u> <u>PDE</u> for h_{11}, h_{12}, h_{22}. But treating such PDEs is beyond the scope of this class.

- A simpler result was shown by <u>Cauchy</u> in the case of polytopes. If P, P' are convex polytopes with the same lengths of edges, angles between edges. Then they are related by a rigid motion.

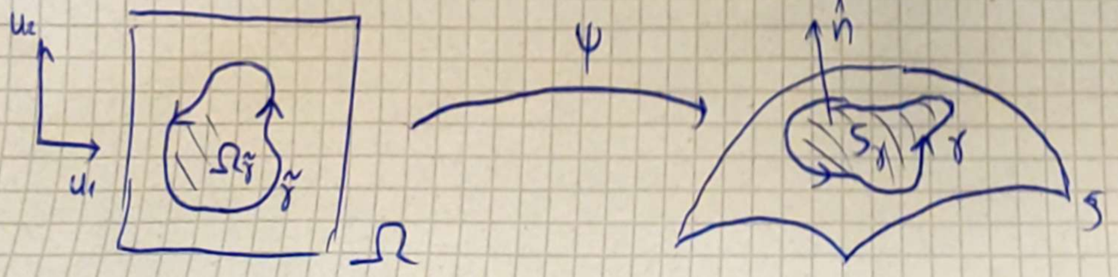
  - Not true for non-convex polytopes: One can construct non-rigid, non-convex polytopes made with metal edges, fixed angles.

<u>The Gauss-Bonnet theorem</u>

There are two versions of this result:
- The local one encodes how the Gauss curvature locally affects the geometry of polygonal shapes on a surface.
- The global one relates the Gauss curvature with the topological class of the surface.

<u>Theorem</u> (Gauss-Bonnet / local)

Let $\psi : \Omega \to S \subset \mathbb{R}^3$ be a C^2 local parametrization, where Ω is simply connected. Let $\tilde{\gamma} : [0, \ell] \to \Omega$ be a simple, positively oriented C^2 closed curve and $\gamma = \psi \circ \tilde{\gamma}$ the corresponding curve on S. Let $\Omega_{\tilde{\gamma}} \subset \Omega$ be the domain enclosed by $\tilde{\gamma}$ and $S_\gamma = \psi(\Omega_\gamma)$.



Then $\displaystyle\int_{S_\gamma} K \, dA + \int_\gamma \kappa_g(s) ds = 2\pi.$

K: Gauss curvature

$\kappa_g(s)$: geodesic curvature of γ

s: Arc length parameter

Note: • If γ is a positively oriented closed curve in the plane ($S = \mathbb{R}^2$)

$\kappa_g = \kappa_{or}$ and $K = 0$, so the above becomes the usual formula $\displaystyle\int_\gamma \kappa_{or} \, ds = 2\pi$

• If γ is a great circle on the sphere (so $\kappa_g = 0$)

then $K = \frac{1}{R^2}$, $\text{Area}(S_\gamma) = 2\pi R^2$,

so the formula checks out.

Proof.

The proof will consist of several steps. We will assume (without loss of generality) the following

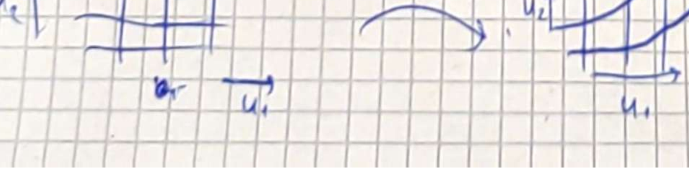
• γ is naturally parametrized

• The parametrization ψ is orthogonal, i.e.

$$G = \begin{pmatrix} g_{11} & 0 \\ 0 & g_{22} \end{pmatrix} \quad (\text{so } b_1 \perp b_2)$$

This can always be arranged by a change of variables: $(u_1, u_2) \longrightarrow (u_1', u_2') = (u_1, u_2 + f(u_1, u_2))$

With f so that $\displaystyle\frac{\partial\psi}{\partial u_1'} = \frac{\partial\psi}{\partial u_1} - \frac{\partial_1 f}{1+\partial_2 f}\frac{\partial\psi}{\partial u_2} \perp \frac{\partial\psi}{\partial u_2'} = \frac{1}{1+\partial_2 f}\frac{\partial\psi}{\partial u_2}$

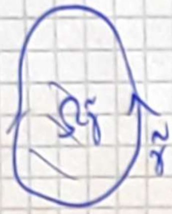


$$\Rightarrow -\partial_1 f + \frac{\langle b_1, b_2\rangle}{\langle b_2, b_2\rangle}\partial_2 f = \frac{\langle b_1, b_2\rangle}{\langle b_2, b_2\rangle}$$

(a transport equation for f that can always be solved)

Lemma 1 (Gauss - Green formula for curl):

If $\tilde{\gamma}(t) = (x(t), y(t))$ is a simple, closed, positively oriented curve in the plane bounding the domain $\Omega_{\tilde{\gamma}}$, then, for every $P, Q : \overline{\Omega_{\gamma}} \to \mathbb{R}$ which are C^1_0:



$$\iint_{\Omega_{\tilde{\gamma}}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy = \int_{\tilde{\gamma}} \left(P \cdot \dot{x}(t) + Q \cdot \dot{y}(t) \right) dt$$

Proof: From Analysis II.

Lemma 2: In an orthogonal parametrization, the surface area form and the Gauss curvature take the form

$$dA = \sqrt{g_{11} \cdot g_{22}}\, du_1\, du_2 \quad \text{and} \quad K = -\frac{1}{2\sqrt{g_{11} g_{22}}} \cdot \left(\partial_1 \left(\frac{\partial_1 g_{22}}{\sqrt{g_{11} g_{22}}} \right) + \partial_2 \left(\frac{\partial_2 g_{11}}{\sqrt{g_{11} g_{22}}} \right) \right)$$

Proof:

The surface area form is $dA = \sqrt{\det G}\, du_1 du_2$ and $\det G = g_{11} \cdot g_{22} - \cancel{g_{12}^2}^{\,0}$.

For the Gauss curvature, recall the Gauss equation.

$$K = \frac{1}{g_{11} g_{22} - g_{12}^2} \left(\partial_1 \Gamma_{221} - \partial_2 \Gamma_{121} - \sum_{l=1}^{2} \left(\Gamma_{11}^l \Gamma_{22l} - \Gamma_{12}^l \Gamma_{12l} \right) \right),$$

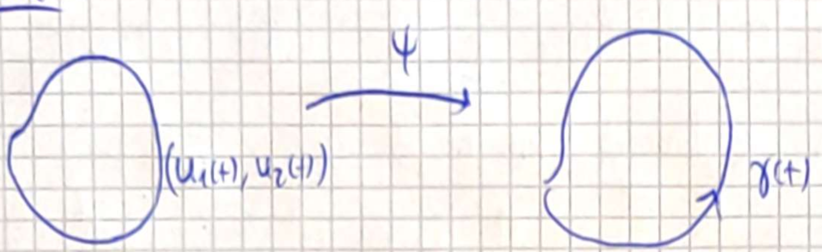
where $\Gamma_{\alpha\beta\gamma} = \frac{1}{2} \left(\partial_\alpha g_{\gamma\beta} + \partial_\beta g_{\gamma\alpha} - \partial_\gamma g_{\alpha\beta} \right)$.

In the case where $g_{12} \equiv 0$, it is a "straightforward" calculation to get the required formula for K ☐

Remark: Note $\iint K\, dA = \iint -\frac{1}{2} \left(\partial_1 \left(\frac{\partial_1 g_{22}}{\sqrt{g_{11} g_{22}}} \right) + \partial_2 \left(\frac{\partial_2 g_{11}}{\sqrt{g_{11} g_{22}}} \right) \right) du_1 du_2$,

so we can apply the Gauss - Green theorem!

Lemma 3



Let $\gamma(s) = \psi(u_1(s), u_2(s))$ be C^2 and parametrized with unit speed. Assume also that the parametrization is orthogonal, then

$$k_g(s) = \frac{1}{2\sqrt{g_{11} g_{22}}} \left(\partial_1 g_{22} \cdot \dot{u}_2 - \partial_2 g_{11} \cdot \dot{u}_1 \right) + \frac{d}{ds}\phi, \qquad \phi = \text{Angle} \langle \dot{\gamma}, b_1 \rangle$$

Proof: Since the parametrization is orthogonal: $\langle b_1, b_2 \rangle = 0$, $\langle b_1, b_1 \rangle = g_{11}$, $\langle b_2, b_2 \rangle = g_{22}$. And $T = \dot{\gamma}(s) = \dot{u}_1 \cdot b_1 + \dot{u}_2 b_2$ $\quad (\|T\| = 1 \Leftrightarrow g_{11} \cdot \dot{u}_1^2 + g_{22} \cdot \dot{u}_2^2 = 1)$

$$\nu = -\sqrt{\frac{g_{22}}{g_{11}}}\, \dot{u}_2 b_1 + \sqrt{\frac{g_{11}}{g_{22}}}\, \dot{u}_1 b_2$$

(Note: $\|\nu\| = 1$, $\nu \perp T$ and $\{T, \nu\}$: positively oriented)

Also: $\displaystyle \frac{d}{dt} b_i(u(t)) = \sum_{j=1}^{2} \frac{\partial b_i}{\partial u_j} \cdot \dot{u}_j = \sum_{j=1}^{2} \frac{\partial^2 \psi}{\partial u_i \partial u_j} \dot{u}_j$

So we compute:

$$\dot{T} = \ddot{u}_1 b_1 + \dot{u}_1 \dot{b}_1 + \ddot{u}_2 b_2 + \dot{u}_2 \dot{b}_2$$

$$= \ddot{u}_1 b_1 + \ddot{u}_2 b_2 + \frac{\partial^2 \psi}{\partial u_1^2} \dot{u}_1 \dot{u}_1 + 2 \frac{\partial^2 \psi}{\partial u_1 \partial u_2} \dot{u}_1 \dot{u}_2 + \frac{\partial^2 \psi}{\partial u_2^2} \dot{u}_2 \dot{u}_2$$

So

$$\kappa_g = \langle \dot{T}, \nu \rangle \overset{b_1 \perp b_2}{=} \langle \ddot{u}_1 b_1 + \ddot{u}_2 b_2, -\sqrt{\frac{g_{22}}{g_{11}}}\, \dot{u}_2 b_1 + \sqrt{\frac{g_{11}}{g_{22}}}\, \dot{u}_1 b_2 \rangle$$

$$+ \langle \sum_{i,j=1}^{2} \frac{\partial^2 \psi}{\partial u_i \partial u_j} \dot{u}_i \dot{u}_j, -\sqrt{\frac{g_{22}}{g_{11}}}\, \dot{u}_2 b_1 + \sqrt{\frac{g_{11}}{g_{22}}}\, \dot{u}_1 b_2 \rangle$$

$\langle b_i, b_j \rangle = g_{ij}$

$\langle \partial^2_{ij} \psi, b_k \rangle = \Gamma_{ijk}$

and $\Gamma^1_{ij} = \frac{1}{\ldots} \cdot \Gamma_{ij1}$ since $g_{12} = 0$

$$= \sqrt{g_{11} g_{22}} \left(-\ddot{u}_1 \dot{u}_2 + \dot{u}_1 \ddot{u}_2 \right) + \sqrt{g_{11} g_{22}} \sum_{i,j=1}^{2} \left(\Gamma^2_{ij} \dot{u}_i \dot{u}_j \dot{u}_1 - \Gamma^1_{ij} \dot{u}_i \dot{u}_j \dot{u}_2 \right)$$

The result now follows from a long computation, using the following ingredients:

- Since $g_{12}=0$, the inverse matrix has components $g^{11}=\frac{1}{g_{11}}$, $g^{22}=\frac{1}{g_{22}}$, $g^{12}=0$

- Since $\Gamma_{ijk} = \frac{1}{2}(\partial_i g_{kj} + \partial_j g_{ki} - \partial_k g_{ij})$, we have

  $$\Gamma_{111} = \frac{1}{2}\partial_1 g_{11}, \quad \Gamma_{112} = -\frac{1}{2}\partial_2 g_{11}, \quad \Gamma_{122} = \frac{1}{2}\partial_1 g_{22}$$

  (and similarly for $1 \leftrightarrow 2$)

- $\dot{\gamma}$ is unit: $g_{11}\dot{u}_1^2 + g_{22}\dot{u}_2^2 = 1$

- $\cos\phi = \frac{\langle \dot{\gamma}, b_1 \rangle}{\|\dot{\gamma}\|\|b_1\|} = \frac{g_{11}\dot{u}_1}{\sqrt{g_{11}}} = \frac{\dot{u}_1}{\sqrt{g_{11}}}$

  and $\sin\phi = \frac{\dot{u}_2}{\sqrt{g_{22}}}$,

- $\dot{g}_{ij} = \frac{d}{ds} g_{ij}(u(s)) = \sum_{k=1}^{2} \partial_k g_{ij} \cdot \dot{u}_k$

We will omit the details of the long calculation $\square$

Putting everything together:

From Lemma 2:

$$\int_{S_\gamma} K\,dA = \int_{\tilde{\Omega}_\gamma} \frac{-1}{2\cancel{\sqrt{g_{11}g_{22}}}}\left(\partial_1\left(\frac{\partial_1 g_{22}}{\sqrt{g_{11}g_{22}}}\right) + \partial_2\left(\frac{\partial_2 g_{11}}{\sqrt{g_{11}g_{22}}}\right)\right)\cancel{\sqrt{g_{11}g_{22}}}\,du_1\,du_2$$

$$= -\int_{\tilde{\Omega}_\gamma}\left(\partial_1\left(\frac{\partial_1 g_{22}}{2\sqrt{g_{11}g_{22}}}\right) - \partial_2\left(-\frac{\partial_2 g_{11}}{2\sqrt{g_{11}g_{22}}}\right)\right) du_1 du_2$$

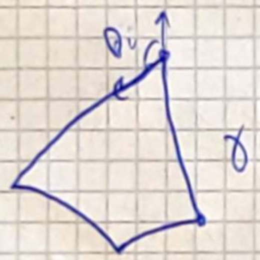
By the Gauss-Green Lemma, this is equal to:

$$= -\int_{\tilde{\gamma}} \left(-\frac{\partial_2 g_{11}}{2\sqrt{g_{11} g_{22}}} \cdot \dot{u}_1 + \frac{\partial_1 g_{22}}{2\sqrt{g_{11} g_{22}}} \dot{u}_2 \right) ds$$

From Lemma 3: $$= -\int_{\gamma} \kappa_g \, ds + \underbrace{\int_{\tilde{\gamma}} \frac{d\phi}{ds} \, ds}_{= 2\pi}$$ (simple closed curve).

So $$\int_{S_\gamma} K \, dA = -\int_{\gamma} \kappa_g \, ds + 2\pi$$

<u>Corollary</u>: If γ is piecewise smooth:



At s_i: $\theta_i = \text{angle}\left(\dot{\gamma}(s_i^-), \dot{\gamma}(s_i^+)\right)$,

then

$$\int_{S_\gamma} K \, dA + \int_{\gamma} \kappa_g \, ds + \sum_{i=1}^{N} \theta_i = 2\pi.$$

Corollary: On a <u>geodesic</u> <u>triangle</u>

$\phi_i = \pi - \theta_i$

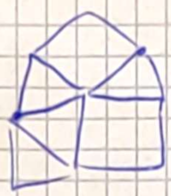
$$\int_{\triangle} K \, dA = \sum_{i=1}^{3} \phi_i - \pi$$

(In the limit of small triangles: this justifies our geometric interpretation of the Gauss curvature as a measure of the deficit of small geodesic triangles)

Gauss-Bonnet theorem (Global)

If S is a compact, C^2 surface:

$$\int_S K \, dA = 2\pi \chi(S), \quad \chi(S): \text{ Euler characteristic of } S$$



If we write S as a "graph", filled in with polygonal faces:

$$\underset{\#\text{ faces}}{\underset{\uparrow}{\chi = F}} - \underset{\#\text{ edges}}{\underset{\uparrow}{E}} + \underset{\#\text{ vertices}}{\underset{\uparrow}{V}}$$

(topological invariant).

So: Two surfaces that are homeomorphic (e.g. two topological spheres): Have the same $\int_S K \, dA$

- Sketch of proof:
  - Make a triangulation of the surface

  - In each triangle: Apply the local Gauss-Bonnet
  - Sum over the triangles, noting that edges belonging to adjacent triangles are traversed in the opposite direction with respect to a positive parametrization for each triangle, so the $\int_\gamma k_g \, ds$ terms cancel out.